The Application of Boundary-Layer Theory to Power-Law Pseudoplastic Fluids: Similar Solutions

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Two- and three-dimensional boundary-layer equations have been developed for pseudoplastic non-Newtonian fluids which can be characterized by a power-law relationship between shear stress and velocity gradient. The types of potential flows necessary for similar solutions to the boundary-layer equations have been determined. For two-dimensional flow the results are similar to those obtained for Newtonian fluids. For three-dimensional flow, however, the possibility of similar solutions depends on the nature of the expression which describes effective viscosity of the fluid. At most, similqr solutions are possible only for the case of flow past a flat plate where the potential velocity vector is not perpendicular to the leading edge of the plate; this is a much more restrictive condition than is obtained for Newtonian fluids.

Most problems in fluid mechanics require a solution of the equation of motion

$$\vec{f} - \rho \frac{D\vec{V}}{Dt} = \nabla \cdot (\vec{\vec{\tau}} + \vec{\vec{I}} p) \qquad (1)$$

The solution of this equation is beset with two difficulties: evaluation of $\vec{\vec{\tau}}$ in terms of known variables and solution of the resulting differential equation. The first difficulty is readily overcome in the case of incompressible Newtonian fluids:

$$\vec{\vec{\tau}} = -\mu \vec{\vec{\Delta}} \qquad (2)$$

where

$$\vec{\vec{\Delta}} = \nabla \vec{V} + (\nabla \vec{V})^{+}$$

and

$$\Delta_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$$

The formulation of $\vec{\vec{\tau}}$ for non-Newtonian fluids is a difficult problem which has not progressed very far from a theoretical standpoint; consequently, several empirical descriptions have been used with varying degrees of success. A number of non-Newtonian fluids can be characterized by a so-called "power-law" description.

$$\frac{1}{\rho} \tau_{xy} = -K \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \qquad (3)$$

When $n > 1$, the fluid is dilatant; $n < 1$ denotes a pseudoplastic fluid. As Reiner (*8*) points out, this is not a law at all but merely an empirical description which in most cases is not entirely adequate. Nevertheless, for want of a more fundamental understanding, Equation (3) has become a well-established device for the description of certain fluids.

Exact solution of Equation (*1*) has been possible for only a few restricted types of flow geometries. In an attempt to give a wider application to Equation (*1*), Prandtl in 1904 introduced the concept of the boundary layer. This concept is based upon the approximation that $\vec{\vec{\tau}}$ is important only in a thin layer near the flow boundaries. The boundary-layer approximations for Newtonian fluids require that

$$\frac{L U_\infty}{\nu} >> 1$$

Boundary-layer theory has been of great value in determining velocity profiles, drag coefficients, and heat transfer coefficients for systems involving the flow of Newtonian fluids past various shapes of solid surfaces (9). The purpose of this paper is to discuss the applicability of boundary-layer theory to the two- and three-dimensional flow of pseudoplastic power-law fluids. Special emphasis is given to the formulation of boundary-layer equations which provide similar solutions.

It should be noted that a solution of the non-Newtonian boundary-layer equation has recently been obtained for the case of two-dimensional flow past a flat plate at zero incidence to the free-stream velocity (*1*).

BOUNDARY-LAYER APPROXIMATIONS

The usual application of Equation (3) is to symmetrical flow geometries where $v = w = 0$ and $\partial/(\partial x) = 0$, x being taken in the direction of flow. However, general boundary-layer flow is more complicated, since one is faced with the problem of expressing all the components of the tensor $\vec{\vec{\tau}}$. It is at this point that the empirical nature of Equation (3) must be remembered. The empirical constants K and n are usually evaluated with one of the standard types of viscometers (5), but there is no *a priori* reason that the indices so evaluated will be the appropriate K and n to use for all the components of $\vec{\vec{\tau}}$ when the fluid is undergoing a general three-dimensional flow. There is a need for experimental study of this problem. In the present analysis it will be assumed that the fluid is isotropic.

Whatever form is chosen for a pseudoplastic power-law description of $\vec{\vec{\tau}}$, it is necessary for $\vec{\vec{\tau}}$ to obey the laws of tensor transformation and for τ_{xy} to have the form given by Equation (3) for simple rectilinear flow. The tensor character of $\vec{\vec{\tau}}$ has been discussed by Oldroyd (7), who suggests that if

$$-\frac{\vec{\vec{\tau}}}{\rho} = \nu_{eff}\,\vec{\vec{\Delta}} \qquad (4)$$

then ν_{eff} can be expressed in terms of the three invariants of $\vec{\vec{\Delta}}$, namely

$$I_1 = 2\,\nabla\cdot\vec{V}$$

$$I_2 = \frac{1}{2}(\vec{\vec{\Delta}}:\vec{\vec{\Delta}}) = \sum_{j=1}^{j=3}\sum_{i=1}^{i=3}\left\{\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \left(\frac{\partial v_j}{\partial x_i}\right)^2\right\}$$

$$I_3 = |\Delta_{ij}|$$

where $|\Delta_{ij}|$ refers to the determinant of $\vec{\vec{\Delta}}$. Since the continuity equation for incompressible fluids

$$\nabla\cdot\vec{V} = 0$$

ensures that I_1 is zero, ν_{eff} will be a function of I_2 and I_3. Hence a possible expression for ν_{eff} in the general case is

$$\nu_{eff} = K\,[(I_2)^{\frac{n-1}{2}} + f_1(n, I_2, I_3)] \quad (5)$$

where f_1 is some function which must be zero for $n = 1$ and/or $I_3 = 0$. These restrictions ensure that $\vec{\vec{\tau}}$ reduces to (2) for a Newtonian fluid and to (3) for simple rectilinear flow, since I_3 is zero for two-dimensional flow. The exact nature of f_1 is not known, and it is of course possible that f_1 is zero for all values of I_3. Substitution of Equation (5) into (4) results in an expression for $\vec{\vec{\tau}}$ which obeys the laws of tensor transformation. When the external force term is neglected, the $\vec{i}$ component of the equation of motion for steady flow becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \vec{i}\cdot(\nabla\cdot\frac{\vec{\vec{\tau}}}{\rho}) \quad (6)$$

The other components are similar. Equation (6) may be rewritten in dimensionless form by referring all lengths to a characteristic length and all velocities to a characteristic velocity U_∞. Thus if $u^\circ = u/U_\infty$, $v^\circ = v/U_\infty$, $x^\circ = x/L$, $p^\circ = p/\rho U_\infty^2$, etc., and the coordinates are chosen as shown in Figure 1

$$u^\circ\frac{\partial u^\circ}{\partial x^\circ} + v^\circ\frac{\partial u^\circ}{\partial y^\circ} + w^\circ\frac{\partial u^\circ}{\partial z^\circ} = -\frac{\partial p^\circ}{\partial x^\circ} + \frac{KU_\infty^{n-2}}{L^n}\,\vec{i}\cdot\{\nabla^\circ\cdot[[(I^\circ_2)^{\frac{n-2}{2}} + f^\circ_1(n, I_2, I_3)]\vec{\vec{\Delta}}^\circ]\} \quad (7)$$

Following Schlichting (9), one can determine the relative magnitudes of the terms in Equation (7), if the usual boundary-layer approximation is made; namely that the dimensionless ratio of the boundary-layer thickness to the characteristic length is small. Hence $\delta << 1$. The characteristic velocity is of the order of magnitude of the velocity outside the boundary layer, and so u° is of order one. Also L is selected so that $(\partial u^\circ)/(\partial x^\circ) \lesseqgtr 1$, and it is assumed that w° and $(\partial w^\circ)/(\partial z^\circ)$ are of an order of magnitude $\lesseqgtr 1$. Then if one defines a Reynolds number $R = L^n/(KU_\infty^{n-2})$ and assigns to it a magnitude R_m, the magnitudes of the terms in Equation (7) are as follow:

Term 1: $u^\circ\dfrac{\partial u^\circ}{\partial x^\circ} \sim 1$

Term 2: $v^\circ\dfrac{\partial u^\circ}{\partial y^\circ} \sim \delta\cdot\dfrac{1}{\delta}$

The quantity v° is of the order δ, since the continuity equation requires $\dfrac{\partial v^\circ}{\partial y^\circ} \sim 1$.

Term 3: $w^\circ\dfrac{\partial u^\circ}{\partial z^\circ} \sim 1$

Term 4: $\dfrac{\partial p^\circ}{\partial x^\circ}$ unknown

Term 5: $\dfrac{1}{R}\dfrac{\partial}{\partial y^\circ}\{[(\dfrac{\partial u^\circ}{\partial y^\circ})^2 + (\dfrac{\partial w^\circ}{\partial y^\circ})^2]^{\frac{n-1}{2}}(\dfrac{\partial u^\circ}{\partial y^\circ})\} + \dfrac{1}{R}f^\circ_{2x} \sim \dfrac{1}{R_m}\dfrac{1}{\delta^{n+1}}$

Only the largest terms have been retained in Term 5 for $n>0$. The quantity f°_{2x}, which denotes the significant terms of the operation $\vec{i}\cdot\nabla^\circ\cdot[f^\circ_1\vec{\vec{\Delta}}^\circ]$, is assumed to be of no greater order of magnitude than the largest terms which result from $\vec{i}\cdot\nabla^\circ\cdot[(I^\circ_2)^{\frac{n-2}{2}}\vec{\vec{\Delta}}]$. To give indentical orders of magnitude to the inertial and friction terms of Equation (7), R_m must be of order $1/\delta^{n+1}$.

From the dimensionless equation of motion in the $\vec{j}$ direction one finds that $(\partial p^\circ)/(\partial y^\circ)$ can be of no greater magnitude than δ. This means that $(\partial p^\circ)/(\partial x^\circ)$ must be of the same order of magnitude within the boundary layer as it is in the potential-flow region outside the boundary layer. Hence the boundary-layer approximation for the $\vec{i}$ component of the equation of motion becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} + K\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial y}\right\} + Kf_{2x} \quad (8)$$

where $U = U(x,z)$ and $W = W(x,z)$. A similar analysis for the $\vec{k}$ component of the equation of motion results in

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} + K\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w}{\partial y}\right\} + Kf_{2z} \quad (9)$$

It is important to recognize that the applicability of Equations (8) and (9) is determined in part by the extent to which the inequality

$$R \cong \frac{1}{\delta^{n+1}} >> 1$$

is true and that R does not depend on the effective viscosity in the same way that it does on the viscosity of a Newtonian fluid. Thus while it is sometimes stated that boundary-layer theory applies to liquids of low viscosity and therefore not to non-Newtonian fluids, which are characterized by a "high" viscosity, it is the magnitude of the Reynolds number which is ultimately the criterion for applicability. For example, for a 33% lime suspension in water flowing under such conditions that L = 1 ft. and U_∞ = 5 ft./sec., using the rheological properties tabulated by Metzner (5) one finds that $R \cong 310$ or $\delta \cong 7 \times 10^{-3}$, thus fulfilling the condition $\delta << 1$.

SIMILAR SOLUTIONS

One of the important considerations of boundary-layer theory is a determination of the types of potential flows for which the boundary-layer equations will possess similar solutions. Solutions which provide expressions for u/U or w/W as functions of a single parameter η, where $\eta = \eta(x, y, z)$, are similar. A study of similar solutions has been made by Hansen and Herzig (3) for incompressible Newtonian fluids in three-dimensional flow and by Schlichting and others (2, 4, 9) for two-dimensional flow. The remainder of this paper is concerned with similar solutions for pseudoplastic non-Newtonian fluids which can be characterized by a power-law relationship such as that of Equation (5). Only similarity with respect to rectangular Cartesian coordinates is considered in the present analysis. It is possible that more general results can be obtained by the use of an approach analogous to that outlined by Morgan (6).

As before, the potential velocity is given by

$$U = U(x, z)$$
$$W = W(x, z)$$

Two functions $F(\eta)$ and $G(\eta)$ are defined so that

$$u = UF'(\eta) \qquad (10)$$

$$w = WG'(\eta) \qquad (11)$$

The prime denotes differentiation with respect to η, where

$$\eta = \frac{y R^N}{Lg(x, z)} \qquad (12)$$

The form of η is analogous to that of the similarity parameter commonly used with Newtonian fluids. It is possible that other similarity parameters could be used (*cf.* reference 3), but they do not appear to have any advantage over Equation (12).

The boundary conditions require that

$$u = v = w = 0 \text{ at } y = 0$$

and

$$\lim_{\eta\to\infty} F' = \lim_{\eta\to\infty} G' = 1$$

Following Hansen and Herzig (3), one can show from the continuity equation that

$$v\left(\frac{R^N}{L}\right) = -g\left(F\frac{\partial U}{\partial x} + G\frac{\partial W}{\partial z}\right) + \frac{\partial g}{\partial x}U(\eta F' - F) + W\frac{\partial g}{\partial z}(\eta G' - G) \qquad (13)$$

where $F(0) = G(0) = 0$.

Application of Equations (10) through (13) to Equation (8) and subsequent conversion to dimensionless quantities result in

$$\frac{g^{n+1}}{(U^\circ)^{n-1}}\frac{\partial U^\circ}{\partial x^\circ}[(F')^2 - FF'' - 1] + \frac{g^{n+1}W^\circ}{U^\circ}\frac{\partial U^\circ}{\partial z^\circ}[F'G' - 1] - \frac{g^{n+1}}{(U^\circ)^{n-1}}GF''\frac{\partial W^\circ}{\partial z^\circ} - \frac{g^n}{(U^\circ)^{n-2}}\frac{\partial g}{\partial x^\circ}FF'' - \frac{g^n W^\circ}{(U^\circ)^{n-1}}F''G\frac{\partial g}{\partial z^\circ} = R^{N(n+1)-1}\frac{\partial}{\partial \eta}\left\{[(F'')^2 + (\frac{W}{U}G'')^2]^{\frac{n-1}{2}}F''\right\} + \frac{KLg^{n+1}}{U_\infty^2(U^\circ)^n}f_{2x} \qquad (14)$$

The explicit Reynolds number dependence is removed by setting $N = 1/(n+1)$. Rearranging terms yields

$$\frac{1}{(U^\circ)^{n-1}}\frac{\partial U^\circ}{\partial x^\circ}[(F')^2 - FF'' - 1] + W^\circ\frac{\partial \ln U^\circ}{\partial z^\circ}[F'G' - 1] - \frac{1}{(U^\circ)^{n-1}}\frac{\partial W^\circ}{\partial z^\circ}GF'' - \frac{1}{(U^\circ)^{n-2}}\frac{\partial \ln g}{\partial x^\circ}FF'' - \frac{W^\circ}{(U^\circ)^{n-1}}\frac{\partial \ln g}{\partial z^\circ}F''G = \frac{1}{g^{n+1}}\frac{\partial}{\partial \eta}\left\{\left[(F'')^2 + \left(\frac{W}{U}G''\right)^2\right]^{\frac{n-1}{2}}F''\right\} + \frac{KL}{U_\infty^2(U^\circ)^n}f_{2x} \qquad (15)$$

A similar result is obtained from Equation (9):

$$\frac{1}{(W^\circ)^{n-1}}\frac{\partial W^\circ}{\partial z^\circ}[(G')^2 - GG'' - 1] + U^\circ\frac{\partial \ln W^\circ}{\partial x^\circ}[F'G' - 1] - \frac{1}{(W^\circ)^{n-1}}\frac{\partial U^\circ}{\partial x^\circ}FG'' - \frac{1}{(W^\circ)^{n-2}}\frac{\partial \ln g}{\partial z^\circ}GG'' - \frac{U^\circ}{(W^\circ)^{n-1}}\frac{\partial \ln g}{\partial x^\circ}FG'' = \frac{1}{g^{n+1}}\frac{\partial}{\partial \eta}\left\{\left[\left(\frac{U}{W}F''\right)^2 + (G'')^2\right]^{\frac{n-1}{2}}G''\right\} + \frac{KL}{U_\infty^2(W^\circ)^n}f_{2z} \qquad (16)$$

If Equations (15) and (16) can be expressed in terms of functions of η alone, the equations will be ordinary differential equations and solutions will provide the form of F' and G' to be used in Equations (10) and (11). Such solutions for u/U or w/W are similar. The author wishes to establish the types of three-dimensional potential flows which will permit elimination of all quantities from Equations (15) and (16) which are not expressible as functions of η alone. An immediate problem is posed by the presence of W/U and U/W in the right sides of Equations (15) and (16) respectively. If the equations are to reduce to ordinary differential equations, it is apparent that one must have W/U = const. unless F'' and G'' are zero. However the latter condition does not permit fulfillment of the boundary conditions for Equations (10) and (11).

If W/U = const. and it is presumed for the moment that f_1 is zero, Equations (15) and (16) can be expressed as ordinary differential equations in η for the case where $n<1$ if

$$(U^\circ)^{1-n}\frac{\partial U^\circ}{\partial x^\circ} = a_1 W^\circ\frac{\partial \ln U^\circ}{\partial z^\circ} = a_2(U^\circ)^{1-n}\frac{\partial W^\circ}{\partial z^\circ} = a_3(U^\circ)^{2-n}\frac{\partial \ln g}{\partial x^\circ} = a_4(U^\circ)^{1-n}W^\circ\frac{\partial \ln g}{\partial z^\circ} = \frac{a_5}{g^{n+1}} = a_6(W^\circ)^{1-n}\frac{\partial W^\circ}{\partial z^\circ} = a_7 U^\circ\frac{\partial \ln W^\circ}{\partial x^\circ} = a_8(W^\circ)^{1-n}\frac{\partial U^\circ}{\partial x^\circ} = a_9(W^\circ)^{2-n}\frac{\partial \ln g}{\partial z^\circ} = a_{10}(W^\circ)^{1-n}U^\circ\frac{\partial \ln g}{\partial x^\circ} \qquad (17)$$

where the a_i represent proportionality coefficients not equal to zero, and it has been assumed that all the terms in Equations (15) and (16) other than those arising from f_1 have nonzero values. The requirement of proportionality between U and W is also stated in Equation (17):

$$(U^\circ)^{1-n} = a_8(W^\circ)^{1-n}$$

Therefore

$$(U^\circ)^{1-n}\frac{\partial U^\circ}{\partial x^\circ} = a_7 U^\circ\frac{\partial \ln\left(U^\circ a_8^{\frac{1}{n-1}}\right)}{\partial x^\circ} = a_7\frac{\partial U^\circ}{\partial x^\circ} \qquad (18)$$

Equation (18) can be satisfied only if $\partial U^\circ/\partial x^\circ = 0$, a condition which is incompatible with the statement that

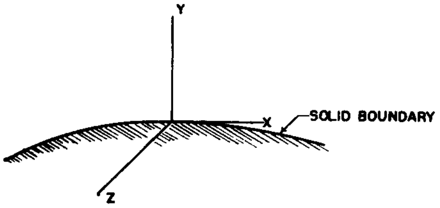

Fig. 1. Orientation of coordinate system with respect to solid surface.

no terms of Equation (17) are zero. A similar result is obtained for $\partial W^\circ/\partial z^\circ$ from the requirement that $a_2(U^\circ)^{1-n}(\partial W^\circ)/(\partial z^\circ) = a_6 (W^\circ)^{1-n} (\partial W^\circ)/(\partial z^\circ)$. Hence for similar solutions

$$\frac{\partial U^\circ}{\partial x^\circ} = \frac{\partial W^\circ}{\partial z^\circ} = 0$$

$$\frac{W^\circ}{U^\circ} = \text{const.}$$

or

$$U^\circ = \text{const.}$$

$$W^\circ = \text{const.}$$

If U° and W° are constant, the similarity condition reduces to

$$a_3(U^\circ)^{2-n}\frac{\partial \ln g}{\partial x^\circ} = a_4(U^\circ)^{1-n} W^\circ \frac{\partial \ln g}{\partial z^\circ} = \frac{a_5}{g^{n+1}} = a_9(W^\circ)^{2-n}\frac{\partial \ln g}{\partial x^\circ} = a_{10}(W^\circ)^{1-n} U^\circ \frac{\partial \ln g}{\partial x^\circ} \quad (19)$$

Equation (19) leads to the result

$$g^{n+1} = a_5(n+1)\left[\frac{1}{a_3}(U^\circ)^{n-2}x^\circ + \frac{1}{a_9}(W^\circ)^{n-2}z^\circ\right] \quad (20)$$

The above analysis shows that similar solutions to the three-dimensional boundary-layer equations can be obtained only for the case of flow past a flat plate, where the potential velocity vector is not parallel to one of the coordinate axes. On the other hand, similar solutions for Newtonian fluids exist for several types of potential flows (3). If the potential velocity vector is perpendicular to the leading edge of the plate, the flow is really two dimensional, but the coordinate axes have been chosen in such a way that $W \neq 0$. For such a flow I_3, and therefore f_1, are zero. However, U° = const. and W° = const. may also describe flow past a flat plate whose leading edge is not perpendicular to the potential velocity vector. In this case f_1 may be nonzero, and additional proportionality conditions may preclude similarity. Thus at most one can expect similar solutions to the three-dimensional boundary-layer equations only for the case of flow past a flat plate where the potential velocity vector is not perpendicular to the leading edge of the plate.

It should be noted that the terms of Equations (15) and (16) could have been collected differently; however, the results would not be affected.

Next the case of two-dimensional flow is considered, where $W^\circ = 0$, $U^\circ = U^\circ(x)$, and $g = g(x)$. Equation (15) reduces to

$$\frac{g^{n+1}}{(U^\circ)^{n-1}}\frac{dU^\circ}{dx^\circ}[1-(F')^2] + \frac{g^n}{(U^\circ)^{n-1}}\frac{d\,(U^\circ g)}{dx^\circ}FF'' + \frac{d}{d\eta}\left\{[(F'')^2]^{\frac{n-1}{2}}F''\right\} = 0 \quad (21)$$

When one sets

$$\alpha = \frac{g^n}{(U^\circ)^{n-1}}\frac{d(U^\circ g)}{dx^\circ}$$

and

$$\beta = \frac{g^{n+1}}{(U^\circ)^{n-1}}\frac{dU^\circ}{dx^\circ}$$

Equation (21) becomes

$$\beta[1-(F')^2] + \alpha FF'' + \frac{d}{d\eta}\left\{[(F'')^2]^{\frac{n-1}{2}}F''\right\} = 0 \quad (22)$$

and has a form identical to that given by Schlichting (9).

If Equation (22) is to possess similar solutions, it is a necessary condition that α and β be constants. Furthermore from the definitions of α and β

$$\frac{d}{dx^\circ}[g^{n+1}(U^\circ)^{2-n}] = \alpha(n+1) + \beta(1-2n) \quad (23)$$

If one stipulates that $\alpha(n+1) + \beta(1-2n) \neq 0$ and that $g^{n+1}(U^\circ)^{2-n} = 0$ at $x = 0$, Equation (23) can be integrated to give

$$g = \{(U^\circ)^{n-2}x^\circ[\alpha(n+1) + \beta(1-2n)]\}^{\frac{1}{n+1}} \quad (24)$$

It is also true that

$$(\alpha-\beta)\frac{d\ln U^\circ}{dx^\circ} = \beta\frac{d\ln g}{dx^\circ} \quad (25)$$

so that

$$(U^\circ)^{\alpha-\beta} = Ag^\beta \quad (26)$$

Combining Equations (24) and (26) leads to the result

$$U^\circ = [A^{n+1}\{x^\circ[\alpha(n+1) + \beta(1-2n)]\}^\beta]^{\frac{1}{\alpha(n+1)+\beta(1-2n)}} \quad (27)$$

Equations (24) and (27) can be simplified by choosing g in such a way that $\alpha = 1$ (when one assumes $\alpha \neq 0$). Such an operation will alter g only by a constant factor and therefore will not interfere with the analysis. If one then defines

$$\gamma = \frac{\beta}{n+1+\beta(1-2n)}$$

$$\beta = \frac{\gamma(n+1)}{1-\gamma(1-2n)} \quad (28)$$

Equations (24) and (27) become

$$g = \left[(U^\circ)^{n-2}\frac{(n+1)x^\circ}{1+\gamma(2n-1)}\right]^{\frac{1}{n+1}} \quad (29)$$

$$U^\circ = A^{1+\gamma(2n-1)}\left[\frac{(n+1)x^\circ}{1+\gamma(2n-1)}\right]^\gamma \quad (30)$$

These two equations show that for the case of $\alpha \neq 0$ and $\alpha(n+1) + \beta(1-2n) \neq 0$ similar solutions are possible whenever the potential velocity is proportional to x° raised to some power. This is exactly the same conclusion which is obtained for Newtonian fluids (9).

If for example a potential flow can be described by

$$U^\circ = a(x^\circ)^c \quad (31)$$

the boundary-layer equation can be transformed by setting

$$g = \left[a^{n-2}(x^\circ)^{c(n-2)+1}\frac{n+1}{1+c(2n-1)}\right]^{\frac{1}{n+1}}$$

Then from Equation (12)

$$\eta = y\left[\frac{1+c(2n-1)}{(n+1)KU_\infty^{n-2}L\,a^{n-2}(x^\circ)^{c(n-2)+1}}\right]^{\frac{1}{n+1}}$$

For the special case where $\alpha = 0$, Equations (24) and (27) reduce to

$$g = [(U^\circ)^{n-2}x^\circ\beta(1-2n)]^{\frac{1}{n+1}} \quad (32)$$

$$U^\circ = [A^{\frac{n+1}{\beta}}x^\circ\beta(1-2n)]^{\frac{1}{1-2n}} \quad (33)$$

In the event that $\alpha \neq 0$ but $\alpha(n+1) + \beta(1-2n) = 0$, one obtains from

Equation (23)

$$g^{n+1}(U^\circ)^{2-n} = B \quad (34)$$

Using the definition of β along with the special constraint which relates α and β, one gets

$$\alpha - \beta = \frac{n-2}{n+1} B \frac{d\ln U^\circ}{dx^\circ} \quad (35)$$

or

$$U^\circ = C\, e^{mx^\circ} \quad (36)$$

where

$$m = \left(\frac{n+1}{n-2}\right) \frac{\alpha-\beta}{B}$$

Equation (36) is identical in form to the result obtained for Newtonian fluids when $2\alpha - \beta = 0$.

From the definitions of α and β

$$\alpha - \beta = g^n (U^\circ)^{2-n} \frac{dg}{dx^\circ} \quad (37)$$

When one substitutes Equation (36) into (37) and integrates,

$$g^{n+1} = \frac{(\alpha-\beta)(n+1)}{C^{2-n} m(n-2)} e^{mx^\circ(n-2)} \quad (38)$$

In this case $g(0)$ has been chosen so that

$$[g(0)]^{n+1} = \frac{(\alpha-\beta)(n+1)}{C^{2-n} m(n-2)}$$

The function g can again be chosen in such a way that $\alpha = 1$; then

$$\beta = \frac{n+1}{2n-1}$$

and

$$g^{n+1} = \frac{(n+1)\, C^{n-2} e^{m(n-2)x^\circ}}{m(2n-1)} \quad (39)$$

Equation (39) gives g in terms of quantities known from the potential velocity and from the physical properties of the fluid.

SUMMARY

The foregoing analysis shows that similar solutions to the boundary-layer equations for pseudoplastic power-law fluids are mathematically possible for certain types of potential-velocity relationships. In two-dimensional flow the potential flows which permit similar solutions are analogous to those which allow similar solutions in the case of Newtonian fluids. In three-dimensional flow the possibility for similar solutions depends on the form of f_1. At most, similar solutions will be possible only for the case of flow past a flat plate where the potential-velocity vector is not perpendicular to the leading edge of the plate. This is a much more restrictive result than is obtained for Newtonian fluids in three-dimensional flow.

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NOTATION

- A = constant of integration in Equation (26)
- a = constant in Equation (31)
- a_i = proportionality constants in Equation (17)
- B = constant of integration in Equation (34)
- C = constant of integration in Equation (36)
- c = constant in Equation (31)
- F = function of η
- f = external force
- f_1 = function defined by Equation (5)
- f_{2x} = largest terms which result from the operation $\vec{i}\cdot\nabla^\circ\cdot[f_1^\circ\vec{\vec{\Delta}}^\circ]$
- f_{2z} = largest terms which result from the operation $\vec{k}\cdot\nabla^\circ\cdot[f_1^\circ\vec{\vec{\Delta}}^\circ]$
- G = function of η
- g = function of x and z
- $\vec{\vec{I}}$ = unit tensor
- I_1, I_2, I_3 = the three invariants associated with $\vec{\vec{\Delta}}$
- $\vec{i}, \vec{j}, \vec{k}$ = unit vectors in the directions shown in Figure 1
- K = kinematic fluid consistency index
- L = characteristic length
- m = $\left(\frac{n+1}{n-2}\right)\frac{\alpha-\beta}{B}$
- N = $\frac{1}{n+1}$
- n = flow-behavior index
- p = pressure
- R = $\frac{L^n}{KU_\infty^{n-2}}$
- R_m = order of magnitude of R
- U, V, W = potential velocity components in the $\vec{i}$, $\vec{j}$, and $\vec{k}$ directions, respectively
- u, v, w = velocity components in the $\vec{i}$, $\vec{j}$, and $\vec{k}$ directions, respectively
- v_i = the i^{th} component of $\vec{V}$
- U_∞ = characteristic velocity which is of the order of magnitude of the potential velocity
- $\vec{V}$ = velocity vector
- x, y, z = distance components in the $\vec{i}$, $\vec{j}$, and $\vec{k}$ directions, respectively
- x_i = x, y, or z for $i = 1$, 2, or 3, respectively

Greek Letters

- α = $g^n(U^\circ)^{1-n}\frac{d(U^\circ g)}{dx^\circ}$
- β = $g^{n+1}(U^\circ)^{1-n}\frac{dU^\circ}{dx^\circ}$
- γ = $\frac{\beta}{n+1+\beta(1-2n)}$
- $\vec{\vec{\Delta}}$ = $\nabla\vec{V} + (\nabla\vec{V})^+$
- Δ_{ij} = $\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$
- δ = dimensionless boundary-layer thickness
- η = $\frac{yR^N}{L\,g(x,z)}$
- μ = viscosity
- ν = kinematic viscosity
- ν_{eff} = effective kinematic viscosity defined by Equation (4)
- $\vec{\vec{\pi}}$ = pressure tensor
- ρ = density
- $\vec{\vec{\tau}}$ = $\vec{\vec{\pi}} - \vec{\vec{I}}p$
- τ_{xy} = component of the pressure tensor on the $\vec{j}$ face of an element of fluid and acting in the $\vec{i}$ direction

Superscripts

- $^\circ$ denotes dimensionless quantity
- $'$ denotes differentiation with respect to η
- $(\nabla\vec{V})^+$ = transpose of $\nabla\vec{V}$

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